Goldstein
9.2

$$\begin{bmatrix} Q \\ P \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix}$$

$$\begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} Q \\ P \end{bmatrix}$$

$$\frac{\partial Q}{\partial q} = \cos\alpha, \quad \frac{\partial p}{\partial P} = \cos\alpha \quad \text{(check)}$$

$$\frac{\partial Q}{\partial p} = -\sin\alpha, \quad -\frac{\partial q}{\partial P} = \sin\alpha \quad \text{(check)}.$$

$$\left(\frac{\partial Q_i}{\partial q_j}\right)_{q,p} = \left(\frac{\partial p_j}{\partial P_i}\right)_{Q,P}, \quad \left(\frac{\partial Q_i}{\partial p_j}\right)_{q,p} = -\left(\frac{\partial q_j}{\partial P_i}\right)_{Q,P}$$

is satisfied, thus the trans. is canonical.

The matrix form of the transformations illustrate the physical significance of this trans. in phase space.

$$Q = \cos\alpha\, q - \sin\alpha\, p$$
$$P = \sin\alpha\, q + \cos\alpha\, p$$

$$q = \cos\alpha\, Q + \sin\alpha\, P$$
$$p = -\sin\alpha\, Q + \cos\alpha\, P$$

$$p = -\sin\alpha\left[\cos\alpha\, q - \sin\alpha\, p\right] + \cos\alpha\, P$$

$$= \sin^2\alpha\, p - \sin\alpha\cos\alpha\, q + \cos\alpha\, P$$

$$(1 - \sin^2\alpha)\, p = \cos\alpha\, P - \sin\alpha\cos\alpha\, q, \quad \boxed{p = \frac{1}{\cos\alpha} P - \frac{\sin\alpha}{\cos\alpha} q}$$

$$Q = \cos\alpha\, q - \sin\alpha\, p$$

$$= \cos\alpha\, q - \sin\alpha\left(-\sin\alpha\, Q + \cos\alpha\, P\right)$$

$$= \sin^2\alpha\, Q - \sin\alpha\cos\alpha\, P + \cos\alpha\, q$$

$$\cancel{1-\sin}\ (1 - \sin^2\alpha)\, Q = \cos\alpha\, q - \sin\alpha\cos\alpha\, P, \quad \boxed{Q = \frac{1}{\cos\alpha} q - \frac{\sin\alpha}{\cos\alpha} P}$$

$$\frac{\partial F_2}{\partial q} = \frac{1}{\cos\alpha} P - \frac{\sin\alpha}{\cos\alpha} q, \qquad \frac{\partial F_2}{\partial p} = \frac{1}{\cos\alpha} q - \frac{\sin\alpha}{\cos\alpha} P$$

$$\boxed{F_2 = \frac{1}{\cos\alpha} Pq - \frac{1}{2}\frac{\sin\alpha}{\cos\alpha} q^2 - \frac{1}{2}\frac{\sin\alpha}{\cos\alpha} P^2}$$

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